2

Projective planes

Möge diese Büchlein dazu beitragen dies schöne Gebiet über die Jahrhunderte zu retten.

Blaschke, Projektive Geometrie 1949

The basis of all investigations in this book will be projective geometry. Although, projective geometry has a tradition of more than 200 years it gives a fresh look at many problems, even today. One could even say that the essence of this book is to view many well known geometric effects/setups/statements/environments from a projective viewpoint.

One of the usual approaches to projective geometry is the axiomatic one. There, in the spirit of Euclid, a few axioms are set up and a projective geometry is definied as any system that satisfies these axioms. We will very briefly meet this approach in this chapter. The main part of this book will, however, be much more concrete and "down to earth". We will predominatly study projective geometries that are defined over a specific coordinate field (most prominently the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$). This gives us the chance to directly investigate the interplay of geometric objects (points, lines, circles, conics,...) and the algebraic structures (coordinates, polynomials, determinants,...) that are used to represent them. Most part of the book will be about surprisingly elegant ways of expressing geometric operations or relations by algebraic formulas. We will in particular focus on understanding the geometry of real and of complex spaces. In the same way as the concept of complex numbers explains many of the seemingly complicated effects for real situations (for instance in calculus, algebra or complex function theory), studying the complex projective world will give surprising insights in the geometry over the real numbers (which to a large extend governs our real life).

The usual study of Euclidean geometry leads to a treatment of special cases at a very early stage. Two lines may intersect or not depending on whether they are parallel or not. Two circles may intersect or not depending on their radii and on the position of their midpoints. In fact, already these two effects lead to a variety of special cases in constructions and theorems all over euclidean geometry. The treatment of these special cases often unnecessary obscures the beauty of the underlying structures. Our aim in this book is to derive statements and formulas that are elegant, general and carry as much geometric information as possible. Here we do not strive for complicated formulas but for formulas that carry much structural insight and often simplicity. In a sense this book is written in the spririt of Julius Plücker (1801–1868) who was as Felix Klein expressed it a master of "reading in the equations".

Starting from the usual Euclidean Plane we will see that there are two essential extensions needed to bypass the special situations described in the last paragraph. First, one has to introduce *elements at infinity.* These elements at infinity will nicely unify special cases that come from parallel situations. Second (in the latter part of this book) we will study the geometry over complex numbers since they allow us to treat also intersections of circles, that are distinct from each other in real space.

2.1 Drawings and perspectives

- *In the Garden of Eden, God is giving Adam a geometry lesson: "Two parallel lines intersect at infinity. It can't be proved but I've been there."*
- *If parallel lines meet at infinity - infinity must be a very noisy place with all those lines crashing together!*

Two math jokes from a website

It was one of the major achievements of the Renaissance period of painting to understand the laws of perspective drawing. If you try to produce a two-dimensional image if a three dimensional object (say a cube or a pyramid), the lines of the drawing cannot be in arbitrary position. Lines that are parallel in the original scene must either be parallel or meet in a point in the picture. Lines that meet in a point in the original scene have either to meet in a point in the drawing or they may become parallel in the picture for very specific choices of the viewpoint. The artists of that time (among others Durer, Da Vinci and Raphael) used these principles to produce (for the standards of that time) stunningly realistic looking images of buildings, towns and other sceneries. The

Fig. 2.1. A page of Durer's book

principles developed at this time still form the bases of most computer created photorealistic images even nowadays. The basic idea is simple. To produce a two-dimensional drawing of a three dimensional scene fix the position of the canvas and the position of the viewers eye in space. For each point on the canvas consider a line from the viewers eye through this point and plot a dot according to the object that your ray meets first (compare Figure 2.1).

By this procedure a line in object space is in general mapped to a line in the picture. One may think of this process in the following way: Any point in object space is connected to the viewpoint by a line. The intersection of this line with the canvas gives the image of the point. For any line in object space we consider the plane spanned by this line and the viewpoint (if the line does not pass through the viewpoint this plane is unique). The intersection of this plane and the canvas plane is the image of the line. This simple construction principle implies that – almost obviously – incidences of points and lines are preserved by the mapping process and that lines are again mapped to lines. Parallelism, orthogonality, distances and angles, however, are not preserved by this process. So it may happen that lines that were parallel in object space are mapped to concurrent lines in the image space. Two pictures in which this construction principles are carried out in a vary strict sense are reproduced in Figure 2.2.

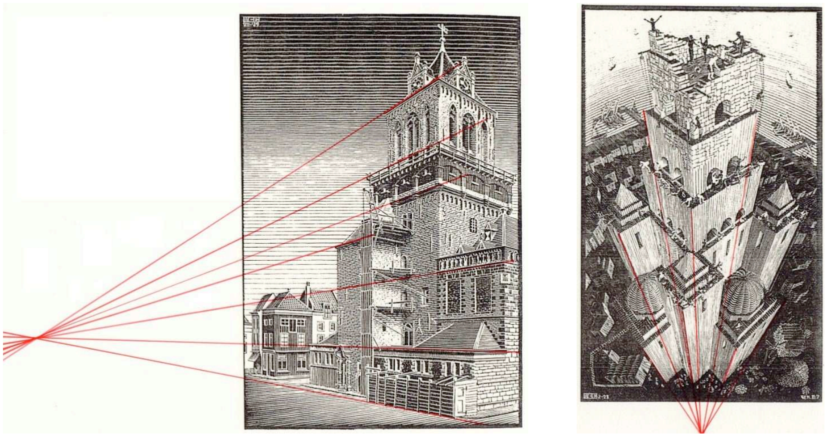

Fig. 2.2. Two copperplates of the dutch graphic artist M.C.Escher

A first systematical treatment of the mathematical laws of perspective drawings was undertaken by the french architect and engineer Girard Desargues (1591 – 1661) and later by his student Blaise Pascal (1623 – 1662). They laid foundations of the discipline that we today call *projective geometry*. Unfortunately many of their geometric investigations have not bee nanticipated by the mathematicians of their time, since approximately at the same time Réne Descartes (1596-1650) published his groundbreaking work *La géométrie* which at the first time intimitely related the concepts of algebra and geometry by introducing a *coordinate system* (this is why we speak of "Cartesian Coordinates"). It was almost 150 later that large parts of projective geometry were rediscovered by the frenchmen Gaspard Monge (1746 – 1818) who was among other duties draftsman, lecturer, minister and a strong supporter of Napolen Bon Aparte and his revolution. His mathematical investigations had very practical backgrounds since they were at least partially directly related to mechanics, architecture and military applications. 1790 Monge wrote a book on what we today would call *constructive* or *descriptive geometry*. This discipline deals with the problem of making exact two-dimensional construction sketches of three dimensional objects. Monge introduced a method (which in essence is still used today by architects or mechanical engineers) of providing different interrelated perspective drawings of a three dimensional object in a predefined way, such that the three dimensional object is uniquely determined by the sketches. Monges method usually projects an object parallel to two or three distinct canvases that are orthogonal to each other. Thus the planar sketch contains, for instance a front view, a side view and as top view of the same object. The line in which the two canvases intersect is identified and

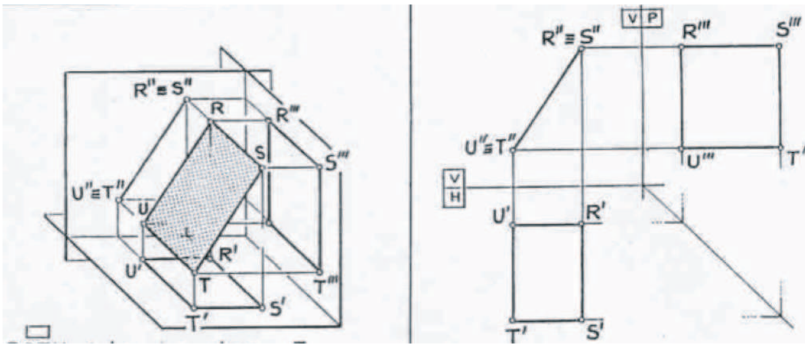

Fig. 2.3. Monge view of a square in space.

commonly used in both perspective drawings. For an example of this method consider Figure 2.3

Monge made the exciting observation that relations between geometric objects in space and their perspective drawings may lead to genuinely planar theorems. These planar theorems can be entirely interpreted in the plane and need no further reference to the original spatial object. For instance consider the triangle in space (see Figure 2.4). Assume that a triangle A, B, C is projected to two different mutually perpendicular projection planes. The vertices of the triangle are mapped to points A', B', C' and A'', B'', C'' in the projection planes. Furthermore assume that the plane that supports the triangle contains the line ℓ in which the two projection planes meet. Under this condition the images ab' and ab'' of the line supporting the edge AB will also intersect in the line ℓ. The same holds for the images ac' and ac'' and for bc' and bc''. Now let us assume that we are trying to construct such a descriptive geometry drawing without reference to the spatial triangle. The fact that ab' and ab'' meet in ℓ can be interpreted as the fact that the spatial line AB meets ℓ. Similarly, the fact that ac' and ac'' meet in ℓ corresponds to the fact that the spatial line AC meets ℓ. However, this already implies that the plane that supports the triangle contains ℓ. Hence, line BC has to meet ℓ as well and therefore bc' and bc'' also will meet in ℓ. Thus the last coincidence in the theorem will occur automatically. In other words, in the drawing the last coincidence of lines occurs automatically. In fact, this special situation is nothing else than Desargues's Theorem that was discovered almost 200 years earlier.

Our starting point, and the last person of our little historical review was Monge's student Jean-Victor Poncelet (1788-1867). He took up Monge's ideas and elaborated on them on a more abstract level. In 1822 he finished his "Traité des propriétés projectives des figures". In this monumental work (about 1200 big foliant pages) he investigated those properties which remain

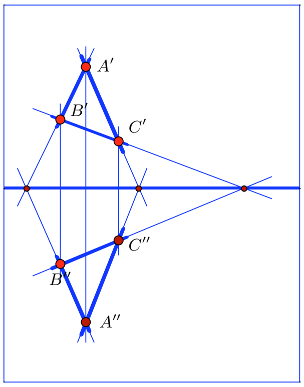

Fig. 2.4. Monge view of a triangle in space

invariant under projection. This two volume book contains fundamental ideas of projective geometry such as the cross-ratio, perspective, involution and the circular points at infinity, that we will meet in many situations troughout the rest of this book. Poncelet was the first one who consequently made use of *elements at infinity* which form the basis of all the elegant treatments that we will encounter later on.

2.2 The axioms

What happens if we try to untangle planar Euclidean Geometry by eliminating special cases arising from parallelism. In Euclidean Geometry two distinct lines intersect unless they are parallel. Now in the setup of projective geometry one enlarges the geometric setup by claiming that two distinct lines will always intersect. Even if they are parallel they have an intersection – we just don't see it. In the axiomatic approach a *Projective Plane* is defined in the following way.

Definition 2.1. *A* projective plane *is a triple* $(\mathcal{P}, \mathcal{L}, \mathcal{I})$*. The set* $\mathcal{P}$ *are the* points*, and the set* $\mathcal{L}$ *are the* lines *of the geometry.* $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ *is an incidence relation satisfying the following three axioms:*

(i) *For any two distinct points, there is exactly one line incident with both of them.*
(ii) *For any two distinct lines, there is exactly one point incident with both of them.*
(iii) *There are four points such that no line is incident with more than two of them.*

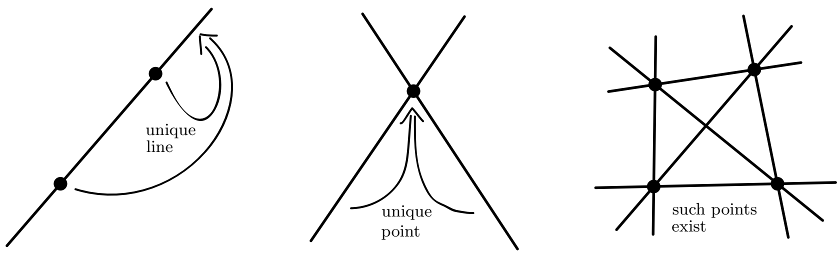

Fig. 2.5. The Axioms of projective geometry.

Observe that the first two axioms describe a completely symmetric relation of points and lines. The second axiom simply states that (without any exception) two distinct lines will always intersect in a unique point. The first axiom states that (without any exception) two distinct points will always have a line joining them. The third axiom merely ensures that the structure is not a degenerate trivial case in which most of the points are collinear.

It is the aim of this and the following section to give various models for this axiom system. Let us first see how the usual Euclidean plane can be extended to a projective plane in a natural way by including elements at infinity. Let $\mathbb{E} = (\mathcal{P}_{\mathbb{E}}, \mathcal{L}_{\mathbb{E}}, \mathcal{I}_{\mathbb{E}})$ be the usual Euclidean plane with points $\mathcal{P}_{\mathbb{E}}$, lines $\mathcal{L}_{\mathbb{E}}$ and the usual incidence relation $\mathcal{L}_{\mathbb{E}}$ of the euclidean plane. We can easily identify $\mathcal{P}_E$ with $\mathbb{R}^2$. Now let us introduce the elements at infinity. For a line l consider the equivalence class $[l]$ of all lines that are parallel to l. For each such equivalence class we define a new point $p_{[l]}$. This point will play the role of the point at infinity in which all the parallels contained in the equivalence class $[l]$ shall meet. This point is supposed to be incident with all lines of $[l]$ Furthermore we define one *line at infinity* l_∞. All points $p_{[l]}$ are supposed to be incident with this line. More formally we set:

- $\mathcal{P} = \mathcal{P}_{\mathbb{E}} \cup \{p_{[l]} \bigm| l \in \mathcal{L}_{\mathbb{E}}\}$,
- $\mathcal{L} = \mathcal{L}_{\mathbb{E}} \cup \{l_\infty\}$,
- $\mathcal{I} = \mathcal{I}_{\mathbb{E}} \cup \{(p_{[l]}, l) \bigm| l \in \mathcal{L}_{\mathbb{E}}\} \cup \{(p_{[l]}, l_\infty) \bigm| l \in \mathcal{L}_{\mathbb{E}}\}$.

It is easy to verify that this system $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ satisfies the axioms of a projective plane. Let us start with axiom (ii). Two distinkt lines l_1 and l_2 have a point in common: If l_1 and $_2$ are non-parallel euclidean lines, then this intersection is simply their usual euclidean intersection. If they are parallel it is the corresponding unique point $p_{[l_1]}$ (which is identical to $p_{[l_2]}$). The intersection of l_∞ with an euclidean line l is the point at infinity $p_{[l]}$ "on" that line. The second axiom is also easy to check: the unique lines incident to two euclidean points p_1 and p_2 is simply the euclidean line between them. The line that joins a euclidean point p and an infinite point p_∞ is the unique line l through p with the property that $p_\infty = p_{[l]}$. Last but not least the line incident to two distinct infinite points is the line at infinity l_∞ itself. This completes the considerations for Axiom (i) and Axiom (ii). Axiom (iii), is evidently satisfied. For this one has simply to pick four points of an arbitrary proper rectangle.

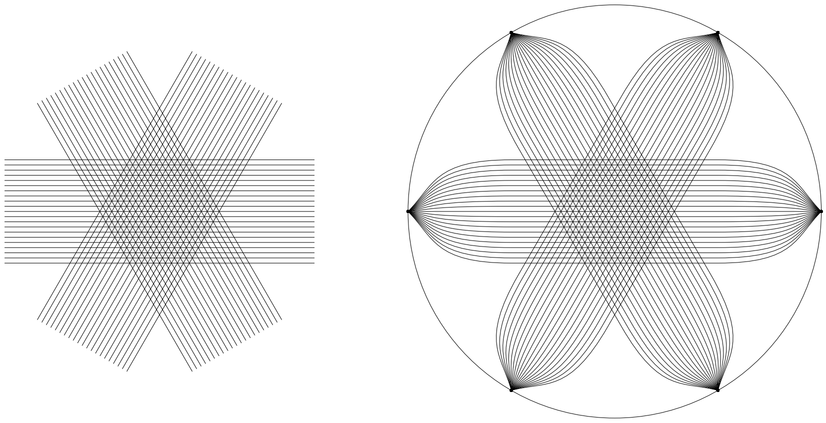

Fig. 2.6. Sketch of some lines in the projective extension of euclidean geometry

Figure 2.6 (left) symbolizes three bundles of parallels in the euclidean plane. Figure 2.6 (right) indicates how these lines projectively meet in a point and how all these points lie together on the line at infinity (drawn as a large circle). Looking at the process of extending the euclidean plane to a projective plane it may seem that the points at infinity and the line at infinity play a special role. We will later on see that this is by far not the case. In a certain sense the projective extension of a euclidean plane is even more symmetric than the usual euclidean plane itself, since it allows for even more automorphisms.

2.3 The smallest projective plane

The concept of projective planes as setup by our three axioms is a very general one. The projective extension of the real euclidean plane is by far not the only model of the axiom system. In fact, still today there is no final classification or enumeration of all possible projective planes. Projective planes do not even have to be infinite objects. There are interesting systems of finitely many points and lines that perfectly satisfy the axioms of a projective plane. To get a feeling for these structures we will briefly construct and encounter a few small examples.

What is the smallest projective plane? Axiom (iii) tells us that it must at least contain four points, no three of which are collinear. So let us start with four points and search for the smallest system of points and lines that contains these points and at the same time satisfies axioms (i) and (ii). Let the four points be A, B, C and D. By axiom (ii) any pair of these points has to be connected by a line. This generates exaclty $\binom{4}{2} = 6$ lines. Axiom (i) requires that any pair of such lines do intersect. There are exacly three missing intersections. Namely those of the pairs of lines $(\overline{AB}, \overline{CD})$, $(\overline{AC}, \overline{BD})$ and $(\overline{AD}, \overline{BC})$. This gives additional three points that must necessarily exist. Now again axiom (i) requires that any pair of points is joined by a line. The only pairs of points that are not joined so far are those formed by the lastly added three points. We can satisfy the axioms by simply adding one line that contains exactly these three points.

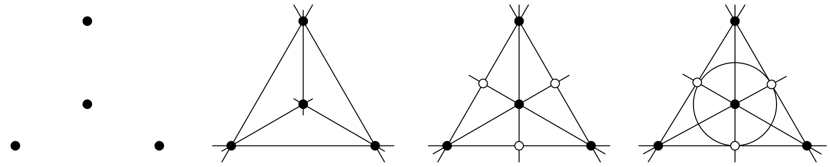

Fig. 2.7. Construction of a small projective plane

The final construction contains seven points and seven lines and is called the *Fano Plane*. There are a few interesting observations that can be made in this example.

- There are exacly as many lines as there are points in the drawing.
- On each line there is exactly the same number of points (here 3).
- Through each point passes exactly the same number of lines.

Each of these statements generalizes to general finite projective planes, as the following propositions show. We first fix some notation. Let $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a projective plane. For a line $l \in \mathcal{L}$ let $p(l) = \{p \in \mathcal{P} \big| p\mathcal{I}l\}$ be the points on l and for a point $p \in \mathcal{P}$ let $l(p) = \{l \in \mathcal{L} \big| p\mathcal{I}l\}$ be the lines through p. Furthermore, we agree on a few linguistic conventions. Since in a projective plane the line l that is at the same time incident to two points p and q is by axiom (i) uniquely determined we will use a more functional rather tham set-theoretic language and simply speek of the *join* of the two points. We will express this join operation by $p \vee q$ or by $\mathbf{join}(p, q)$. Similarly, we will call the unique point incident with two lines l and m the *meet* or *intersection* of these lines and denote the corresponding operation by $l \wedge m$ or by $\mathbf{meet}(l, m)$. We also say sat a line l *contains* a point p if it is incident with it.

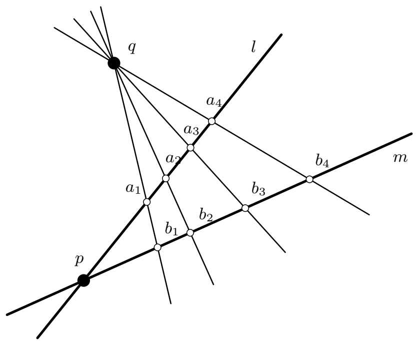

Fig. 2.8. The proof that all lines have the same number of points.

Lemma 2.1. *If for* $p, q \in \mathcal{P}$ *and* $l, m \in \mathcal{L}$ *we have* $p\mathcal{I}l$, $q\mathcal{I}l$, $p\mathcal{I}m$ *and* $q\mathcal{I}m$ *then either* $p = q$ *or* $l = m$.

Proof. Assume that $p\mathcal{I}l$, $q\mathcal{I}l$, $p\mathcal{I}m$ and $q\mathcal{I}m$. If $p \neq q$ axiom (i) implies that $l = m$. □

Lemma 2.2. *Every line of a projective plane is incident with at least three points.*

Proof. Let $l \in \mathcal{L}$ be any line of the projective plane and assume on the contrary that l does contain less than three points. Let a, b, c and d be the points of Axiom (iii). Assume w.l.o.g. that a and b are not on l. Consider the lines $a \vee b$, $a \vee c$, $a \vee d$. Since these all pass through a they must be distinct by axiom (iii) and must by Lemma 2.1 have three distinct intersections with l. □

Lemma 2.3. *For every point* p *there is at least one line not incident with* p.

Proof. Let p be any point. Let l and m be arbitrary lines. Either one if them is does not contain p (then we are done), or we have $p = l \wedge m$. By the last lemma there is a point p_l on l distict from p, and a point p_m on m distinct from p. The join of these two points cannot contain p since this would violate axiom (i). □

Theorem 2.1. *Let* $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ *be a projective plane with finite sets* $\mathcal{P}$ *and* $\mathcal{L}$. *Then there exists a number* $n \in \mathbb{N}$ *such that* $|p(l)| = n + 1$ *for any* $l \in \mathcal{L}$ *and* $|l(p)| = n + 1$ *for any* $p \in \mathcal{P}$.

Proof. Let l and m be two distinct lines. Assume that l contains k points. We will prove that both lines contain the same number of points. Let $p = l \wedge m$

be their intersection and let ℓ be a line through p distinct from l and m. Now consider a point q on ℓ distinct from p, which exists by Lemma 2. Let $\{a_1, a_2, \ldots, a_n\} = p(l) - \{p\}$ be the points on l distinct from p and consider the $n-1$ lines lines $l_i = p_i \vee q; \quad i = 1, \ldots, n$. Each of these lines intersects the line m in a point $b_i = l_i \wedge m$. All these points have to be distinct, since otherwise there would be lines l_i, l_j that intersect twice in contradicion to Lemma 1. Thus the number of points on m is as least as big as the number of points on l. Similarly, we can argue that the number of points on l is as least as big as the number of points on m. Hence both numbers have to be equal. Thus the number of points on a line is the same for any line (see Figure 2.3).

Now let p be any point and l be a line that does not contain p. Let $\{p_1, p_2, \ldots, p_n\}$ be the n points on l. Joining these points with p generates k lines through p. In fact, this must be all lines through p since any line through p must have an intersection with l by axiom (ii). Hence the number of lines that pass through our (arbitrarily chosen) point p must also be equal to k. □

The number n of the last proposition (which was the number of points on a line minus one) is usually called the *order* of the projective plane. The following proposition relates the order and the overall number of points and lines in a finite projective plane.

Theorem 2.2. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a projective plane with finite sets $\mathcal{P}$ and $\mathcal{L}$ of order n. Then we have $|\mathcal{P}| = |\mathcal{L}| = n^2 + n + 1$.*

Proof. The last proposition proved that the number of points on each line is $n+1$ and the number of lines through each point is also $n+1$. Let p be any point of the projective plane. Each of the $n+1$ lines through p contains n additional points. They must all be distinct, since otherwise two of these lines intersect twice. We have alltogether $(n+1) \cdot n + 1 = n^2 + n + 1$ points. A similar count proves that the number of lines is the same. □

So far we know two examples of a projective plane. One is the finite Fano Plane of order 2, the other (infinite example) was the projective extension of the real numbers. Our next chapter will show, that both can be considered as special examples of a construction that generates a projective plane for every number field.

3

Homogeneous coordinates

3.1 A spatial point of view

Let $\mathbb{K}$ be any field[1]. And let $\mathbb{K}^3$ the vector space of dimension three over this field. We will prove that if we consider the one dimensional subspaces of $\mathbb{K}^3$ as *points* and the two dimensional subspaces as *lines*, then we obtain a projective plane by defining *incidence* as subspace containment.

We will prove this fact by creating a more concrete coordinate representation of the one- and two-dimensional subspaces of $\mathbb{K}^3$. This will allow us to be able to calculate with these objects easily. For this we first form equivalence classes of vectors by identifying all vectors $v \in \mathbb{K}^3$ that differ by a non-zero multiple:

$$[v] := \{v' \in \mathbb{K}^3 \mid v' = \lambda \cdot v \text{ for } \lambda \in \mathbb{K} \setminus \{0\}\}.$$

The set of all such equivalence relations could be denoted $\frac{\mathbb{K}^3 \setminus \{(0,0,0)\}}{\mathbb{K} \setminus \{0\}}$; all non-vero vectors modulo scalar non-zero multiples. Replacing a vector by its equivalence class preserves many interesting structural properties. In particular, two vectors v_1, v_2 are orthogonal if their scalar product vanishes: $\langle v_1, v_2 \rangle = 0$. This relation remains stable if we replace the two vectors by any vectors taken from the corresponding equivalence classes. We define orthogonality of equivalence classes $[p]$ and $[l]$ in a canonic way by

$$[p] \perp [l] \quad \Longleftrightarrow \quad \langle p, l \rangle = 0.$$

Now we set $\mathcal{P}_{\mathbb{K}} = \frac{\mathbb{K}^3 \setminus \{(0,0,0)\}}{\mathbb{K} \setminus \{0\}}$ and let $\mathcal{L}_{\mathbb{K}} = \frac{\mathbb{K}^3 \setminus \{(0,0,0)\}}{\mathbb{K} \setminus \{0\}}$ as well (we consider $\mathcal{P}_{\mathbb{K}}$ and $\mathcal{L}_{\mathbb{K}}$ as disjoint copies of the same kind of space). Furthermore we define the incidence relation $\mathcal{I}_{\mathbb{K}} \subseteq \mathcal{P}_{\mathbb{K}} \times \mathcal{L}_{\mathbb{K}}$ for $[p] \in \mathcal{P}$ and $[l] \in \mathcal{L}$ by

$$p\, \mathcal{I}_{\mathbb{K}}\, l \quad \Longleftrightarrow \quad [p] \perp [l].$$

[1] This is almost the only place in this book where we will refer to an arbitrary field $\mathbb{K}$. All other considerations will be much more "down to earth and refer to specific fields" – mostly the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$

Before we prove that the triple $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \mathcal{I}_{\mathbb{K}})$ is indeed a projective plane, we clarify what this has to do with one- and two-dimensional subspaces. There is a bijection of the set of one dimensional subspaces of $\mathbb{K}$ and $\mathcal{P}_{\mathbb{K}}$. Each subspace can be represented by a single non-zero vector p in it. In fact, exactly all vectors in the equivalence class $[p]$ represent the same one-dimensional subspace. $[p]$ itself is this subspace with the zero vector taken out. A two dimensional dimensional vector space $\{(x, y, z) \mid ax + by + cz = 0\}$ in $\mathbb{K}^3$ is in our setup represented by its normal vector (a, b, c). Since normal vectors that differ only by a scalar multiple describe the same two-dimensional subspace the set $\mathcal{L}_{\mathbb{K}}$ is appropriate for representing them. Finally, a one-dimensional subspace represented by $[p]$ is contained in a two-dimensional subspace represented by $[l]$ if and only if $[p] \perp [l]$. This is consistent with our incidence operator $\mathcal{I}_{\mathbb{K}}$.

Theorem 3.1. *With the above definitions and notations for anly field $\mathbb{K}$ the triple $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \mathcal{I}_{\mathbb{K}})$ is a projective plane.*

Proof. We simply have to verify the three axioms. Let $[p]$ and $[q]$ be two distinct elements in $\mathcal{P}_{\mathbb{K}}$. In order to verify axiom (i) we must prove that there is a vector l that is simultaneously orthogonal to p and q. Furthermore we must show that all non-zero vectors with this property must be scalar multiples of l. Since $[p]$ and $[q]$ are distinct the vectors (p_1, p_2, p_3) and (q_1, q_2, q_3) do not differ just by a non-zero scalar multiple. In other words the matrix

$$\begin{pmatrix} p_1 & p_2 & p_4 \\ q_1 & q_2 & q_4 \end{pmatrix}$$

has rank 2. Thus the solution space of

$$\begin{pmatrix} p_1 & p_2 & p_4 \\ q_1 & q_2 & q_4 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is one dimenssional. This is exactly the desired claim. For any non-zero solution (l_1, l_2, l_3) of this system the equivalence class $[(l_1, l_2, l_3)]$ is the desired join of the points.

In a completely similar way, we can verify axiom (ii), which states that for any pair of distinct lines there is exactly one point incident to both.

For axiom (iii) observe that any field $\mathbb{K}$ must contain a zero and a one element. It is easy to check, that the equivalence classes of the four vectors $(0, 0, 1)$, $(0, 1, 1)$,$(1, 0, 1)$ and $(1, 1, 1)$ satisfy the requirements of non-collinearity of axiom (iii). □

Although, the message of the last theorem is simple it is perhaps the central point of this entire book. It is the link of geometry and algebra. It's power stems from the fact that we can recover our construction of projectively extending the Euclidean plane directly in the representation of points by three dimensional vectors. This will be shown in the next section. This representation of points as well as lines of a projective plane by three dimensional vectors is called homogeneous coordinates. We will later on see that the adjective *homogeneous* is very appropriate, since these coordinates at the same time unify the role of usual lines and the line at infinity and give three coordinates of $\mathbb{K}^3$ a completely symmetric interpertation. We will see that by introducing this coordinate system we can easily deal with the Euclidean plane and its projective extension (the points and the line at infinity) in a completely algebraic manner.

The use of homogeneous coordinates can be considered as an externsion of so called *barycentric coordinates*, which were introduced by August Ferdinand Möbius (1790-1869). Homogeneous coordinates were first introduced by Julius Plücker in his article "Ueber ein neues Coordinatensystem" in 1829. There he writes

> *Ich habe bei den folgenden Entwicklungen nur die Absicht gehabt [...] zu zeigen, dass die neue Methode [...] zum Beweise einzelner Sätze und zur Darstellung allgemeiner Theorien sich sehr geschmeidig zeigt.*[2]

In fact it is this elegance that we will use throughout this book and we hope that the reader finally after finishing this book will agree on this.

3.2 The real projective plane with homogeneous coordinates

Let us now investigate how the projective extension of the Euclidean plane fits into the picture of homogeneous coordinates. For this we start with a coordinate representation of the Euclidean plane $\mathbb{E}$. As usual we identify the Euclidean plane with $\mathbb{R}^2$. Each point in the Euclidean plane can be represented by a two dimensional vector of the form $(x, y) \in \mathbb{R}^2$. A line can be considered as the set of all points (x, y) satisfying the equation $a \cdot x + b \cdot y + c = 0$. However, since we will treat lines as individual objects rather than sets of points we will consider the parameters (a, b, c) themselves as a representation of the line. Observe that for non-zero λ the vector $(\lambda \cdot a, \lambda \cdot b, \lambda \cdot c)$ represents the same line as (a, b, c). Furthermore the vector $(0, 0, 1)$ does not represent a real line at all, since then the above equation would read as $1 = 0$.

Now we make the step to homogeneous coordinates. For this we consider our Euclidean plane embedded affinely in the three-dimensional space $\mathbb{R}^3$. It is convenient to consider the plane to be the $z = 1$ plane. Each point (x, y) of the Euclidean plane will now be represented by the point $(x, y, 1)$. How should we interpret all other points in $\mathbb{R}^3$? In fact, for any point that does

[2] Me intention for making the following developements was to demonstrate that this new method turns out to be very pliable for proving specific theorems or for representing general theories.

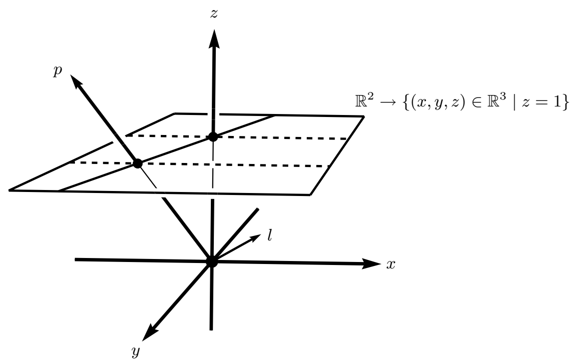

Fig. 3.1. Embedding the Euclidean plane in $\mathbb{R}^3$.

not have a zero z-component we can easily assign a corresponding Euclidean point. For $(x, y, z) \in \mathbb{R}^3$ we consider the one dimensional subspace spanned by this point. If $z \neq 0$ this subspace intersects the embedded Euclidean plane at a unique single point. We can calculate this point simply by dividing by the z coordinate. Thus for $z \neq 0$ the vector $p = (x, y, z)$ represents the Euclidean point $(x/z, y/z, 1)$. Note that all vectors in the equivalence class $[p]$ represent the same Euclidean point — so, if we are only interested in Euclidean points, we de not have to care about non-zero scalar factors.

How about the remaining points of $\mathbb{R}^3$, those with z-coordinate equal to 0? These points will correspond to the *points at infinity* of the projective completion of the Euclidean plane. To see this we consider a limit process that dynamically moves a point to infinity and observe what will happen with the Euclidean coordinates. We start in the Euclidean picture. Assume we have a point $p = (p_1, p_2)$ in the usual Euclidean plane. Furthermore we have a direction (r_1, r_2). If we consider $q_\alpha := p + \alpha \cdot r$ and start to increase α from 0 to a larger and larger value the point q_α will move away in direction r. How does this situation look like in homogeneous coordinates? Point q_α is represented by the homogeneous coordinates $(p_1 + \alpha \cdot r_1, p_2 + \alpha \cdot r_2, 1)$. Since in homogeneous coordinates we do not care about non-zero multiples we can (for $\alpha \neq 0$) equivalently represent the point q_α by $(p_1/\alpha + r_1, p_2/\alpha + r_2, 1/\alpha)$. What happens in the limit case $\alpha \to \infty$? In this case our vector representing q_α degenerates to the vector $(r_1, r_2, 0)$. Let us reinterpret this process geometrically. "No matter with which point we start, if we move it in direction r further and further out then, in the limit case, we will end up at a point with homogeneous coordinates $(r_1, r_2, 0)$." In other words, we can consider the vector $(r_1, r_2, 0)$ as a representation of the point at infinity in direction r. (Perhaps it is a good exercise for the reader to convince himself/herself that we arrive at exactly the same point if we decrease α starting at $\alpha = 0$ and ending at $\alpha = -\infty$. Also for infinite points it is possible to neglect scalar multiples and take any point of the corresponding equivalence class $[(r_1, r_2, 0)]$ to represent the same point at infinity.

The only vector that does not fit to our consideration so far is the zero vector $(0, 0, 0)$. This is, however, no problem at all since the space $\frac{\mathbb{R}^3 \setminus \{(0,0,0)\}}{\mathbb{R} \setminus \{0\}}$ does exclude this vector explicitely. We will later on see, that whenever the zero-vector pops up in a calculation we will have encountered a degenerate situation, for instance intersecting two identical lines.

How about the lines? We already saw that a Euclidean line is nicely represented by the parameters (a, b, c) of the line equation $a \cdot x + b \cdot y + c = 0$. We also observed that multiplying (a, b, c) by a non-zero scaler does not change the line represented. If we view the line equation in homogeneous coordinates it becomes

$$a \cdot x + b \cdot y + c \cdot z \quad = \quad 0.$$

If we consider a point on this line with homogeneous coordinates $(x, y, 1)$ this form degenerates to the Euclidean version. However whenever we have a point (x, y, z) that satisisfies the equation it will still satisfy the equation if we replace it by $(\lambda x, \lambda y, \lambda z)$. Thus, this form is stable under our representation of points and lines by equivalence classes. If we interpret this equation in three dimensions, we see that the vector (a, b, c) is the normal vector of the plane that contains all vectors $(x, y, z) \in \mathbb{R}^3$ that satisfy the equation. If we intersect this plane with our embedded Euclidean plane we obtain a line in the Euclidean plane that corresponds to the Euclidean counterpart of our line under consideration (compare Fig. 3.1).

There is only one type of vector that does not correspond to a Euclidean line. If we consider the vector $(0, 0, c)$ with $c \neq 0$ the orthogonal vector space is the xy-plane through the origin. This plane does not intersect the embedded Euclidean plane. However all points at infinity (remember, they have the form $(x, y, 0)$) are orthogonal to this vector since $0 \cdot x + 0 \cdot y + c \cdot 0 = 0$. We call this line the *line at infinity.* It is incident to all points at infinity.

Let us summarize what we have achieved so far. In Section 2.2 we discussed how we can extend the Euclidean plane by introducing elements at infinity: one point at infinity for each direction and one global line at infinity that contains all these points. Now, we have a concrete coordinate representation of these objects. The Euclidean points correspond to points of the form $(x, y, 1)$, the infinite points correspond to points of the form $(x, y, 0)$. The Euclidean lines have the form (a, b, c) with $a \neq 0$ or $b \neq 0$ (or both). The line at infinity has the form $(0, 0, 1)$. All the vectors are considered modulo non-zero scalar multiples. We will refer to this this setup of the real projective plane later on as $\mathbb{RP}^2$. This notion stands for $\mathbb{R}$eal $\mathbb{P}$rojective 2-dimensional space. Later on we will also get to spaces like $\mathbb{RP}^1$, $\mathbb{RP}^d$, $\mathbb{CP}^1$, $\mathbb{CP}^2$.

Form the three dimensional viewpoint the distinction of infinite and finite elements is completely unnatural: all elements are just represented by vectors. This resembles the situation in the axiom system for projective planes. There we also do not distinguish between finite and infinite elements. This distinction is only a kind of artifact that arises when we interpret the Euclidean plane in a projective setup. In a sense if we consider the projective plane as an extension of the Euclidean plane we break the nice symmetry of projective planes by (artificially) singling out one line to play the role of the line at infinity. Nevertheless, it is a very fruitful exercise to interpret Euclidean theorems in a projective framework or to interpret projective theorems in a Euclidean framework. Usually, a whole group of theorems in Euclidean geometry corresponds to just one theorem in projective geometry and turns out to be just different specializations for different lines at infinity. We will make these kinds of investigations very often in the following chapters and we will see how nicely projective geometry generalizes different Euclidean concepts.

3.3 Joins and meets

This section is dedicated to a way of easily carrying out elementary operations in geometry by algebraic calculation. In Chapter 2 we saw that the axiom system for projective planes immediately motivates two operations the *join of two points* and the *meet of two lines*. We will now get to know the algebraic counterparts of these operations. From now on we will (by slight abuse of notation) no longer explicitly refer to the equivalence classes of points that arise from multiplication with non-zero scalars. Rather than that we will do the calculations with explicit representatives of these classes. Essentially all operations that will be described can be simply carried out on this level of representatives. So, form now on the reader should always have in mind that the vectors (x, y, z) and $(\lambda x, \lambda y, \lambda z)$ represent the same geometric point.

The crucial point for representing the join and meet operations algebraically is that if (in homogeneous coordinates) the point (x, y, z) is contained in the line (a, b, c) the equation

$$a \cdot x + b \cdot y + c \cdot z \quad = \quad 0$$

holds. If the equation holds, then these two vectors are orthogonal. Now, if two points $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ are given, then the coordinates $l = (l_1, l_2, l_3)$ of a line incident to both points must be orthogonal to both vectors p and q. In Section 3.1 we argued that there is a solution to this problem by explicitly writing down a system of linear two linear equations. However, there is also a way to obtain a specific solution explicitly. For this consider the vector-product operator "$\times$ from linear algebra. This operator is defined as follows:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \times \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad = \quad \begin{pmatrix} +p_2 q_3 - p_3 q_2 \\ -p_1 q_3 + p_3 q_1 \\ +p_1 q_2 - p_2 q_1 \end{pmatrix}.$$

An easy calculation shows that this operator generates a vector that is simultaneously orthogonal to p and q. For instance for p we get after term expansion:

$$p_1 \cdot (p_2q_3 - p_3q_2) + p_2 \cdot (-p_1q_3 + p_3q_1) + p_3 \cdot (p_1q_2 - p_2q_1) = 0.$$

(We will soon see a more structural approach to the vector product, that explains this relation.) Thus we can express the join operation of two points simply by the cross product:

$$\mathbf{meet}(p, q) \quad := \quad p \times q.$$

We can deal in a completely similarly fashion with the problem of intersection two lines $l = (l_1, l_2, l_3)$ and $m = (m_1, m_2, m_3)$. A point that is simultaneously incident with both lines must be represented by a vector that is orthogonal to both l and m. We can generate such a vector simply by forming the vector product. Thus we get:

$$\mathbf{join}(l, m) \quad := \quad l \times m.$$

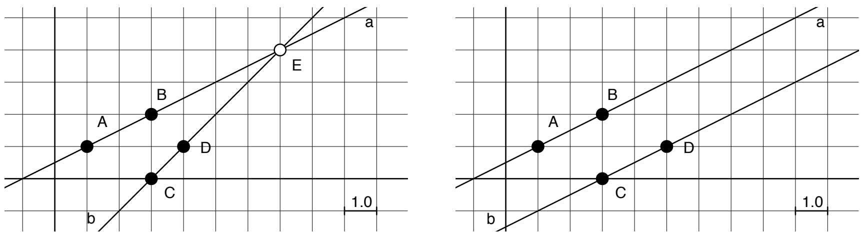

Fig. 3.2. Working with meet and join.

It is instructive to see these operators in work in an Euclidean example. Let A, B, C, D be four points in the Euclidean plane given by the following (Euclidean) coordinates:

$$A = (1, 1),$$
$$B = (3, 2),$$
$$C = (3, 0),$$
$$D = (4, 1).$$

What are the coordinates of the intersection of the lines AB and CD? The homogeneous coordinates of the points are $A = (1, 1, 1), B = (3, 2, 1), C = (3, 0, 1), D = (4, 1, 1)$. We can calculate the homogeneous coordinates of the two lines simply by taking the vector products:

$$\begin{aligned} l_{AB} &= (1,1,1) \times (3,2,1) \\ &= (1 \cdot 1 - 1 \cdot 2 - 1 \cdot 1 + 1 \cdot 3 + 1 \cdot 2 - 1 \cdot 3) \\ &= (-1, 2, -1), \end{aligned}$$

$$\begin{aligned} l_{CD} &= (3,0,1) \times (4,1,1) \\ &= (0 \cdot 1 - 1 \cdot 1 - 3 \cdot 1 + 1 \cdot 4 + 3 \cdot 1 - 0 \cdot 4) \\ &= (-1, 1, 3). \end{aligned}$$

The meet E of these lines is again calculated by the vector products:

$$\begin{aligned} E &= (-1,2,-1) \times (-1,1,3) \\ &= (2 \cdot 3 - (-1) \cdot 1 - (-1) \cdot 3 + (-1) \cdot (-1) + (-1) \cdot 1 - 2 \cdot (-1)) \\ &= (7, 4, 1). \end{aligned}$$

These are the homogeneous coordinates of the Euclidean point $(7,4)$ (The fact that the z-coordinate turned out to be 1 was, in fact, only a lucky coincidence. In general we would have to divide by this coordinate to get the Euclidean values). It is somehow amazing that with a projective point of view we get an explicit and straightforward way to calculate with joins and intersections. The calculations even take automatically care of the coordinates, if elements at infinity are involved. We consider the same example but now with point D located at $(5,1)$. The calculation above becomes:

$$\begin{aligned} l_{AB} &= (1,1,1) \times (3,2,1) \\ &= (1 \cdot 1 - 1 \cdot 2 - 1 \cdot 1 + 1 \cdot 3 + 1 \cdot 2 - 1 \cdot 3) \\ &= (-1, 2, -1), \end{aligned}$$

$$\begin{aligned} l_{CD} &= (3,0,1) \times (5,1,1) \\ &= (0 \cdot 1 - 1 \cdot 1 - 3 \cdot 1 + 1 \cdot 5 + 3 \cdot 1 - 0 \cdot 5) \\ &= (-1, 2, 3). \end{aligned}$$

$$\begin{aligned} E &= (-1,2,-1) \times (-1,2,3) \\ &= (2 \cdot 3 - (-1) \cdot 2 - (-1) \cdot 3 + (-1) \cdot (-1) + (-1) \cdot 2 - 2 \cdot (-1)) \\ &= (8, 4, 0). \end{aligned}$$

Point E is now an infinite point since its z-coordinate is zero. in particular it its the infinite point in direction $(8,4)$ (or equivalently in direction $(2,1)$). This is the point in which the two parallel lines meet.

3.4 Parallelism

The only operations and relations we modeled so far are incidence, join and meet. We will see that many other geometric operations (like measuring distances, calculating angles, creating perpendiculars) will require special treatment if we want to model them in a projective setup. Nevertheless there is at least one operation of Euclidean geometry that can be easily modeled in a projective framework: *Drawing a parallel to a line through a point.* For this start with the real projective plane with our usual setup in homogeneous coordinates. We the have to single out a line at infinity. Usually we use the standard line at infinity with homogenenous coordinates $(0, 0, 1)$, but we are not forced to do so.

Let $l_\infty \in \mathcal{L}_\mathbb{R}$ be the line at infinity. With respect to this line we can define an operator $\mathbf{parallel}(p, l) : \mathcal{P}_\mathbb{R} \times \mathcal{L}_\mathbb{R} \to \mathcal{L}_\mathbb{R}$ that takes as input a line l and a point p and calculates a line parallel to l and through p. We define this operator by:

$$\mathbf{parallel}(p, l) := \mathbf{join}(p, \mathbf{meet}(l, l_\infty)) = p \times (l \times l_\infty).$$

How does this operator work. First it calculates the intersection of l with the line at infinity. This is the point at infinity that is contained in l and on any parallel to l. So, if we want to obtain a parallel to l through p, we have simply to join this point with p. This is how the operator works.

It is interesting to see what happens if we select a finite Euclidean line as the line at infinity. As an example consider the situation of a square and the task of constructing its two diagonals, its center, and two lines through this center which are parallel to the quadrangles sides (Fig. 3.3). If we had chosen four arbitrary (non-square) points A, B, C, D as corners the construction could still be performed. For this let in cyclic order be A, B, C, D the corners of the quadrangle. The joins $d_1 = \mathbf{join}(A, C)$ and $d_2 = \mathbf{join}(B, D)$ are the diagonals of the "quadrangle". Their meet $m = \mathbf{meet}(d_1, d_2)$ is the center. To get the two parallels we first have to know where the line at infinity is. If we consider (by definition) the four points as corners of a square, we know that opposite sides must be parallel. Hence the intersections of the lines supporting opposite sides gives us two ways of constructing a points at infinity. Namely $p_1 = \mathbf{meet}(\mathbf{join}(A, B), \mathbf{join}(C, D))$ and $p_2 = \mathbf{meet}(\mathbf{join}(B, C), \mathbf{join}(D, A))$. Joining these two points gives us the position of the line at infinity. We finally want to construct the two lines through the center, parallel to the sides. This are simply the joins $\mathbf{join}(m, p_1)$ and $\mathbf{join}(m, p_2)$. What we finally obtain is a perspectively correct drawing of the quadrangle together with the required points and lines.

3.5 Duality

We will here briefly touch a topic, that we will encounter later in greater depth and detail. You may have observed, that if we are in a projective setup points and lines play a completely symmetric role. We want to point out a few points where this becomes transparent.

- In the axiom system for projective planes axiom (i) transferes to axiom (ii) if one interchanges the words line and point.

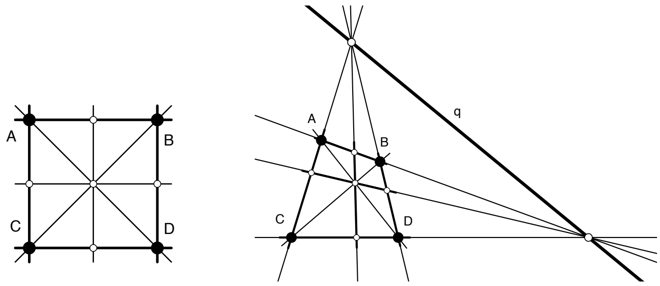

Fig. 3.3. Working with meet and join

- At first sight, axiom (iii) seems to break symmetry, however one can proof a similar statement with the role of points and lines interchanged as a consequence of the three axioms.
- In the homogeneous coordinate setup the spaces $\mathcal{P}_{\mathbb{K}}$ and $\mathcal{L}_{\mathbb{K}}$ are algebraically identical.
- In the incidence relation $ax + by + cz = 0$ the vectors (a, b, c) and (x, y, z) play a completely symmetric role.
- *Joins* and *meets* can both be calculated by the vector product.

So, every statement in projective geometry that only involves the vocabulary we developed so far is again transferred to a true statement if we exchange the terms:

$$\begin{aligned} \text{point} &\leftrightarrow \text{line} \\ \text{join} &\leftrightarrow \text{meet} \\ \mathcal{P} &\leftrightarrow \mathcal{L} \end{aligned}$$

We call this effect *duality*. So we can say that very basis of projective geometry is dual. This implies that for every concept we will develop further on there will be a corresponding dual counterpart. For every theorem in projective geometry there will be a corresponding dual theorem. For every definition in projective geometry there will be a corresponding dual definition, and so forth. The reader is invited to dualize the rest of this book (i.e. it is useful to question for every concept/theorem/definition/drawing introduced in the book what would be the corresponding dual).

We will exemplify duality with a small construction of projective geometry (compare Fig. 3.4). We first describe the primal construction. We start with four points of which no three are collinear in $\mathbb{RP}^2$. There are all together six lines that can be drawn between these four points. Dually this reads: Start with four lines. These lines will have all together six points of intersection. The pictures of the primal and the dual situation are drawn in the picture above.

One has to be aware that the analogy of primal and dual situations goes far beyond the combinatorial level. We can literally take the homogeneous

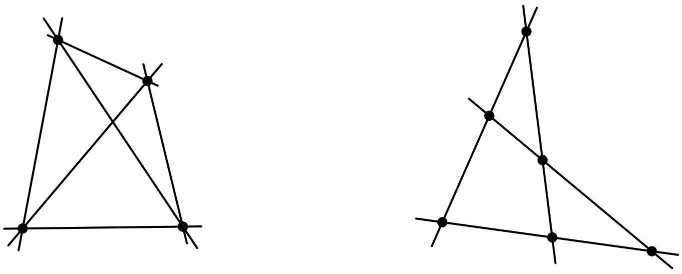

Fig. 3.4. A pair of primal and dual configuration.

coordinates of a point and interpret them as homogeneous coordinates of a line, and vice versa. Incidences are preserved under this exchange. Figure 3.5 represents an example of three collinear points in the standard embedding of the Euclidean plane on the $z = 1$ plane. Coordinates of the points and of the line are given. The second picture shows the corresponding dual situation in which the coordinates are interpreted as line coordinates. Three lines that meet in a point. The line equations are given and it is easy to check that the homogeneous coordinates of the points in one picture are exactly the homogeneous coordinates of the lines in the other picture.

3.6 Projective transformations

Transformations are a fundamental concept all over geometry. There are different aspects under which one can consider transformation. On the one hand they are a change of the frame of reference. The same objects are after a transformation represented within a new coordinate system. Hence a transformation is a (bijective) map of the ambient space onto itself. The other way one can look at transformations is that they take the objects and move (or even deform) them to end up in another position. No matter which picture

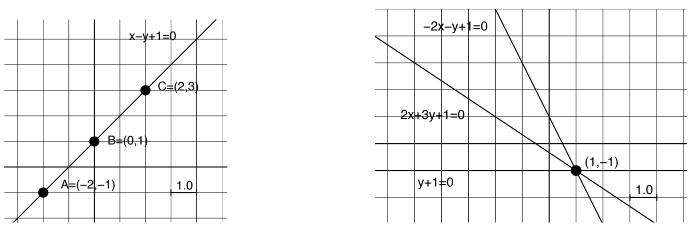

Fig. 3.5. A pair of primal and dual configuration with coordinates.

one prefers to describe a transformation, the crucial point is that they leave certain properties of the objects unchanged.

We will first introduce transformations in an abstract setup and become more and more specific further on. In general on can equip reasonable collections of transformations with a group structure. For this let us consider an object space $\mathbb{O}$. This object space will later on be, for instance, the set of points $\mathcal{P}_{\mathbb{R}}$ of the real projective plane. In general a transformation is a bijective map $T: \mathbb{O} \to \mathbb{O}$. We obtain the group structure by requiring that collections of transformations should be closed under reasonable operations. If one applies two transformations T_1 and T_2 one after another one can consider the result as a single transformation $(T_2 \circ T_1): \mathbb{O} \to \mathbb{O}$. For this book we make the convention that $T_2 \circ T_1$ is interpreted as first applying T_1 and then T_2. Thus if we have a specific object $o \in \mathbb{O}$ we have $(T_2 \circ T_1)(o) = T_2(T_1(o))$. The identity $Id: \mathbb{O} \to \mathbb{O}$ that maps every element of the object space to itself is a transformation. Since transformations are assumed to be bijective maps in the object space, we can for any transformation T consider its inverse operation T^{-1} as a transformation as well. We have $T \circ T^{-1} = Id$. It is also not difficult to check that transformations are in general associative. For this we have to show, that if we have three transformations T_1, T_2, T_3 the relation $(T_3 \circ T_2) \circ T_1 = T_3 \circ (T_2 \circ T_1)$ holds. In order to see this consider a concrete object o. We have

$$\begin{aligned}((T_3 \circ T_2) \circ T_1)(o) &= (T_3 \circ T_2)(T_1(o)) \\ &= T_3(T_2(T_1(o)) \\ &= T_3((T_2 \circ T_1)(o)) \\ &= (T_3 \circ (T_2 \circ T_1))(o).\end{aligned}$$

Taking all this together one obtains a the properties that ensure that we have a *group structure*.

Let us be a little more concrete and consider the usual transformations of Euclidean geometry (we will now recall a few facts from linear algebra). For this let again $\mathbb{R}^2$ represent the coordinates of the Euclidean plane. The points of the Euclidean plane will be our objects, thus $\mathbb{R}^2$ plays the role of the object space. The usual transformations in Euclidean geometry are *translations*, *rotations*, *reflections* and *glide reflections*. These transformations can easily expressed by algebraic operations. A translation by a vector (t_x, t_y) can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + t_x \\ y + t_y \end{pmatrix}.$$

A rotation about the origin by an angle α can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

A rotation about an arbitrary point (r_x, r_y) can be written as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} x - r_x \\ y - r_y \end{pmatrix} + \begin{pmatrix} r_x \\ r_y \end{pmatrix}.$$

Reflections and glide reflections have a similar representation. Any of the above Euclidean transformations can be written in the form

$$p \mapsto M(p - v) + w$$

for suitable choices of a 2×2 matrix M and vectors v and w. For rotations The matrix M has to be a rotation matrix. This means it has the form $\begin{pmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{pmatrix}$. For reflections or glide reflections the matrix must be a reflection matrix of the form $\begin{pmatrix} cos(\alpha) & sin(\alpha) \\ sin(\alpha) & -cos(\alpha) \end{pmatrix}$. The group of Euclidean transformations leaves fundamental properties and relations within the object space invariant. For instance, if p and q are Euclidean points, then their distance is the same before or after an Euclidean transformation. Also the absolute value of angles are not altered by Euclidean transformations. In general the *shape* and *size* of an object is not altered by a Euclidean transformation. If one point-by-point maps a circle (or line, or quadrangle) by a Euclidean transformation one ends up again with a circle (or line, or quadrangle) of the same size. It may have just have moved to another location.

In the above form $M(p - v) + w$ one may allow for more general transformations (where M is *any* invertible 2×2 matrix). By this one can also describe scalings, similarities or affine transformations. In this case the group of transformations becomes larger and the set of properties that is not altered by this transformations becomes smaller. For instance similarities will still preserve the absolute value of angles but no longer distances. An affine transformation will not even preserve angles. However, an affine transformation still maps a pair of parallel lines to another pair of parallel lines.

From the point of view of computer implementations it is inherently difficult and error prone to calculate with the above representation of Euclidean transformations. The fact that the rotational or reflectional part is expressed by a matrix multiplication while the translational part is expressed by a vector addition makes it cumbersome to calculate the inverses or the succession of two transformations. Again we get a structurally much clearer approach if we focus on a projective setup and an approach via homogeneous coordinates.

If we represent an Euclidean point (x, y) by homogeneous coordinates $(x, y, 1)$ we can express rotations as well as translations by a multiplication with a 3×3 matrix. Translations take the following form (assuming for a moment that the z-coordinate is chosen to be 1):

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}.$$

Rotations about the origin can be expressed as:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} cos(\alpha) & sin(\alpha) & 0 \\ -sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Applying two transformations in succession is now nothing else but multiplication of the corresponding matrices. Inverting a transformation corresponds to matrix inversion. One should notice that the above matrices were chosen in a way that a vector with z-coordinate equal to one is again mapped to a vector with z-coordinate equal to one. Hence, the first two entries of the homogeneous coordinate vector directly show the Euclidean position of the mapped point (in our standard embedding). From a conceptual point of view, it is, even id one only deals with Euclidean Transformations, often much more useful to work in this more general representation, since here translations, rotations and reflections arise in a unified way. Moreover, we will gain even more advantage from this representation, since it is the key to an even wider class of transformations: the projetive transformations. First, if we consider matrices with non-zero determinant of the following form

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix},$$

then we get all affine planar transformations. Still we have not used the whole freedom of an invertible three by three matrix. A general *projective transformation* is a multiplication by an invertible 3×3 matrix:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

We now want to investigate the properties of such a general type of projective transformation. We first make a notational convention. Since for any $p \in \mathbb{R}^3 \setminus \{0\}$ the product of a 3×3 matrix M with any member of the equivalence class $[p]$ ends up within the same equivalence class $[M \cdot p]$, the action of M on these equivalence classes is well defined. Thus we can simply interpret M as acting on our object space (of equivalence classes) $\mathcal{P}_{\mathbb{R}}$. Thus we can interpret the multiplication by M on the level of representatives taken from $\mathbb{R}^3 \setminus \{(0,0,0)\}$ or on the level of equivalence classes $\frac{\mathbb{R}^3\setminus\{(0,0,0)\}}{\mathbb{R}\setminus\{0\}}$. Thus for a projective point in $[p] \in \mathcal{P}_{\mathbb{R}}$ we will write $M \cdot [p]$ and mean by this the projective point $[M \cdot p]$.

Since in the context of projective geometry the input vector as well as the output vector of our matrix-multiplication are only determined up to a multiplication by a non-zero scalar the matrices M and λM represent the same projective transformation (for non zero λ). Thus we have overall only eight degrees of freedom that determine such a transformation.

One fundamental property of projective transformations is given by the following statement.

Theorem 3.2. *A projective transformation maps collinear points to collinear points.*

Proof. It suffices tho show the theorem for a generic triple of points. Let $[a], [b], [c] \in \mathcal{P}_{\mathbb{R}}$ be three collinear points represented by homogeneous coordinates a, b, c. In this case there exists a line $[l] \in \mathcal{L}_{\mathbb{R}}$ with $\langle l, a\rangle = \langle l, b\rangle = \langle l, c\rangle = 0$. We assume that all homogeneous coordinates are represented by column vectors. We have to show that under these conditions the points represented by $a' = M \cdot a$, $b' = M \cdot b$, $c' = M \cdot c$, are also collinear. For this simply consider the line $[l']$ represented by by $l' := (M^{-1})^T l$. We have:

$$\langle l', a'\rangle = (l')^T a' = ((M^{-1})^T \cdot l)^T \cdot M \cdot a = l^T \cdot ((M^{-1})^T)^T \cdot M \cdot a = l^T \cdot a = \langle l, a\rangle = 0.$$

A similar calculation applies also to the other two points. Thus the line represented by l' is simultaneously incident to all three points represented by a', b' and c'. Hence these points are collinear. □

Implicitly, the last proof describes how a projective transformation $M: \mathcal{P}_{\mathbb{R}} \to \mathcal{P}_{\mathbb{R}}$ represented by a 3×3-matrix M acts on the space of lines $\mathcal{L}_{\mathbb{R}}$. The homogeneous coordinates of a line must be mapped in a way such that incidences of points and lines are preserved under the mapping. This implies that a line has to be mapped according to $l \mapsto (M^{-1})^T l$. If p and l are incident before a transformation they will be incident after the transformation as well.

In fact the property of Theorem 3.2 is characterizing for projective transformation sover the field of real numbers. One can prove:

Theorem 3.3. *If $\Phi: \mathcal{P}_{\mathbb{R}} \to \mathcal{P}_{\mathbb{R}}$ is any bijective map that preserves the collinearity of points, then Φ can be expressed as multiplication by a 3×3 matrix.*

In fact, this theorem is so crucial that it is sometimes called the *fundamental theorem of projective geometry.* Its proof is a little subtle, and requires some elementary results from field theory. The proof makes use of the fact that the real numbers do not have any field automorphisms except the identity. The generalization of the above theorem to arbitrary fields involves a proper discussion of field automorphims. A proof will be postponed to Section 5 when we will discuss the relations of projective geometry and elementary arithmetic operations. For now, we will collect more properties of projective transformations that can be expressed as multiplication by a 3×3 matrix.

The most fundamental property of projective transformations which we will need (which is also of invaluable practical importance) is the following fact.

Theorem 3.4. *Let $[a], [b], [c], [d] \in \mathcal{P}_{\mathbb{R}}$ be four points of which no three are collinear and let $[a'], [b'], [c'], [d'] \in \mathcal{P}_{\mathbb{R}}$ be another four points of which no three are collinear, then there exists a 3×3 matrix M such that $[M \cdot a] = [a']$, $[M \cdot b] = [b']$, $[M \cdot c] = [c']$ and $[M \cdot d] = [d']$.*

Proof. We assume that $a, b, c, d, a', b', c', d' \in \mathbb{R}^3$ are representatives of the corresponding equivalence classes. We first proof the theorem for the special case that $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0, 0, 1)$ and $d = (1, 1, 1)$. Since the columns of a matrix are the images of the unit vectors, the matrix must have the form $(\lambda \cdot a', \mu \cdot b', \tau \cdot c')$. (In other words the image of a must be a multiple of vector a' and so forth.) Hence the image of d is $\lambda \cdot a' + \mu \cdot b' + \tau \cdot c'$. This must be a multiple of d'. We only have to adjust the parameters λ, μ, τ accordingly. For this we have to solve the system of linear equations:

$$\begin{pmatrix} | & | & | \\ a' & b' & c' \\ | & | & | \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \\ \tau \end{pmatrix} = \begin{pmatrix} | \\ d' \\ | \end{pmatrix}.$$

This system is solvable, by our non-degeneracy assumptions (a', b', c' are not collinear). Furthermore none of the parameters is zero (as a consequence of the remaining non-degeneracy assumptions). This proves the theorem for the special case.

In order to prove the general case of the theorem one uses the above fact to find a transformation T_1 that maps $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ to a, b, c, d, and to find a transformation T_2 that maps $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ to a', b', c', d'. The desired transformation is then $T_2 \cdot T_1^- 1$. □

Remark 3.1. (A note on implementations): The last theorem is not only of theoretical interest. The proof gives also a practical recipe for calculating a projective transformation that maps a, b, c, d to a', b', c', d' (as usual up to scalar multiple). The basic operations that are required for this are matrix multiplication and matrix inversion. One has to simply follow the different calculations steps in the above proof.

The fact that projective transformations preserve collinearities and incidences of points and lines relates them intimately to the topic of of perspectively correct drawings. Figure 3.6 shows a drawing of a checker board like grid and four circles and its image under a projective transformation. The projectively transformed picture is completely determined already by the image of four corner points. Observe that for instance the grid points along the diagonals are again collinear in the transformed image. One can also see that angles and distances are not preserved under a projective transformation. Not even ratios of distances are preserved: an equi-distant chain of points in the original picture will in general no longer be equi-distant after the projective transformation (later on we will see that so called *cross-ratios* are preserved under projective transformations). We also see that circles are not necessarily mapped to circles again. The picture also indicates that tangentiality relations of curves are preserved under projective transformations.

Throughout the entire book we will very often come back to the topic of projective transformations under various aspects.

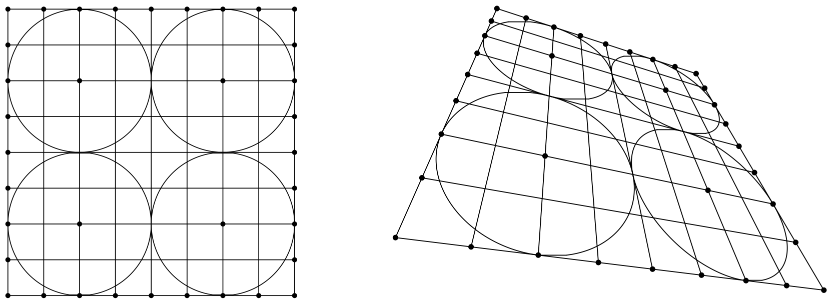

Fig. 3.6. The image of a grid under a projective transformation.

3.7 Finite projective planes

Before we will continue our study of geometric situations over the real (and over the complex) numbers we will have a very brief look at projective spaces over finite fields. Without providing proofs we will report on a few basic facts. The construction of Section 3.1 was a general method of constructing a projective plane starting from a field $\mathbb{K}$. Points correspond to one-dimensional subspaces, lines correspond to two dimensional subspaces. If $\mathbb{K}$ is a finite field we end up with a projective plane consisting of only finitely many points and lines. Let us consider the smallest cases explicitly. First we study the case $\mathbb{K} = GF_2$ the field of characteristic 2 that consists of a 0 and a 1, only. All non-zero vectors of $\mathbb{K}^3$ are listed below:

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Over this field there are no non-trivial scalar multiples of these vectors (the only non-zero scalar is $\lambda = 1$). Hence each of these vectors corresponds to one point of the corresponding projective plane. These seven points are nothing but the seven points of the Fano plane that we encountered in Section 2.3. An assignment of coordinates to the points is given in Figure 3.7. Three points are collinear in this plane if and only if there is a line vector (a, b, c) that is simultaneously orthogonal to all three points. For instance the circle in the center corresponds to the line $(1, 1, 1)$.

Alternatively one can view the Fano plane in the following way: The GF_2 analogue of the Euclidean plane $\mathbb{R}^2$ is the space $(GF_2)^2$ which has exactly four elements. We can homogenize them by embedding them in the $z = 1$ plane of $(GF_2)^3$ (white points in the picture). In addition we have to consider all *points at infinity* with a z coordinate 0 (the black points). They lie on a common line

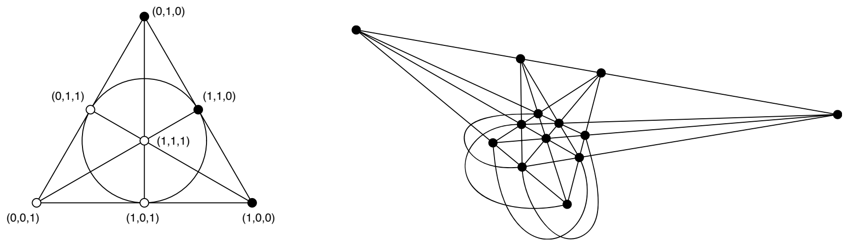

Fig. 3.7. The Fano plane with coordinates over GF_2. And the projective plane over GF_3

– the line at infinity. Observe that each projective line contains exactly $n+1$ elements. We can calculate the number of points in two different ways. If n is the number of elements of the field then we have n^2 finite points and $n+1$ infinite points. This makes all together n^2+n+1 points. We obtain the same number if we consider the (n^3-1) non-zero vetors in $\mathbb{K}^3$. Each equivalence class consists of $n-1$ vectors. And we have $(n^3-1)/(n-1) = n^2+n+1$ points.

The next more complicated example is a projective plane over the three-element field GF_3. Here we have 4 points on each line and an overall number of points is $3^2+3+1=13$. The corresponding incidence structure is shown in Fig. 3.7 (right), also here we could divide the points into a finite and an infinite part and single out a line at infinity. However, one should be aware that by construction there are no a priory distinguished lines: As in the case of the Euclidean plane *any* line can play the role of the line at infinity.

Since there is a finite field for every prime power p our general construction immediately yields the following result:

Theorem 3.5. *For any prime power n there is a projective plane that consists of n^2+n+1 points and n^2+n+1 lines. Each line contains exactly $n+1$ points and each points lies on exactly $n+1$ lines.*

The parameter n is called the order of the finite projective plane. There is a famous conjecture that the order of a projective plane is always a prime power. However experts in the field have tried to prove this conjecture now since several decades and the status of the conjecture remains still open. We briefly want to review the state of this conjecture. A priory there is no reason why for $n>1$ there should not be a projective plane of order n. The sharpest result that rules out several cases is the Theorem of Bruck and Ryser which was first proved in 1949 (which we quote without proof here).

Theorem 3.6. *If a projective plane of order n exists, and $n = 1$ or $2 \pmod 4$, then n is the sum of two squares.*

Let us see what the situation looks like for orders up to 14:

$2 = 2^1 = 1 + 1$	Fano plane;
$3 = 3^1$	Plane over GF_3;
$4 = 2^2$	Plane over $(GF_2)^2$;
$5 = 5^1 = 4 + 1$	Plane over GF_5;
6	Not sum of two squares – no projective plane of this order;
$7 = 7^1$	Plane over GF_7;
$8 = 2^3$	Plane over $(GF_2)^3$;
$9 = 3^2 = 9 + 0$	Plane over $(GF_3)^2$;
$10 = 9 + 1$	No prime power, but Bruck-Ryser does also not apply;
$11 = 11^1$	Plane over GF_{11};
12	No prime power, but Bruck-Ryser does also not apply;
$13 = 13^1$	Plane over GF_{13};
14	Not sum of two squares – no projective plane of this order;

The table onveils two interesting values of the order where neither the Theorem of Bruck and Ryser rules out the existence of a porjetive plane nor our field construction applies: The orders 10 and 12.

The case of order 10 was settled in 1989 by H.W.C. Lam, L. Thiel and S. Swiercz. They proved the non-existance of a projective plane of order 10 by a clever but in essence still brute-force computer proof. The exhaustive computer proof took the equivalent of 2000 hours on a Cray 1 supercomputer. (In order to get an impression of the problem state it in the following way: "Fill a cross table with 111×111 entries such that the following conditions are true. In each row and each column there are exactly 11 crosses. Furthermore each pair of rows must have exactly one cross in the same column.)

The case of order 12 is still widely open. No method seems to be known to break down the difficulty of enumerating all possible cases to a reasonable size that would fit on contemporary computing devices.

One might wonder whether the only way to obtain a finite projective plane is via our field construction. This is not the case. The first case where such non-standard planes occur is order nine. There are 4 non-isomorphic projective planes of this order. There are even 193 (known) finite projective planes of order 25. A general method of classification seems to be far beyond reach.